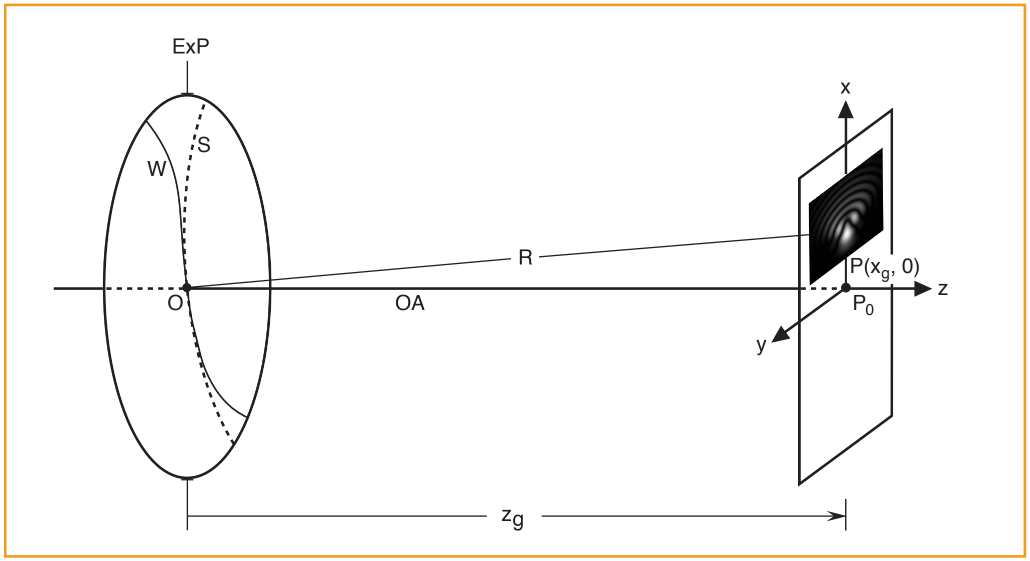
Beam Focusing and Depth of Focus



© Virendra N. Mahajan

The Aerospace Corporation
El Segundo, California 90245
(310) 336-1783

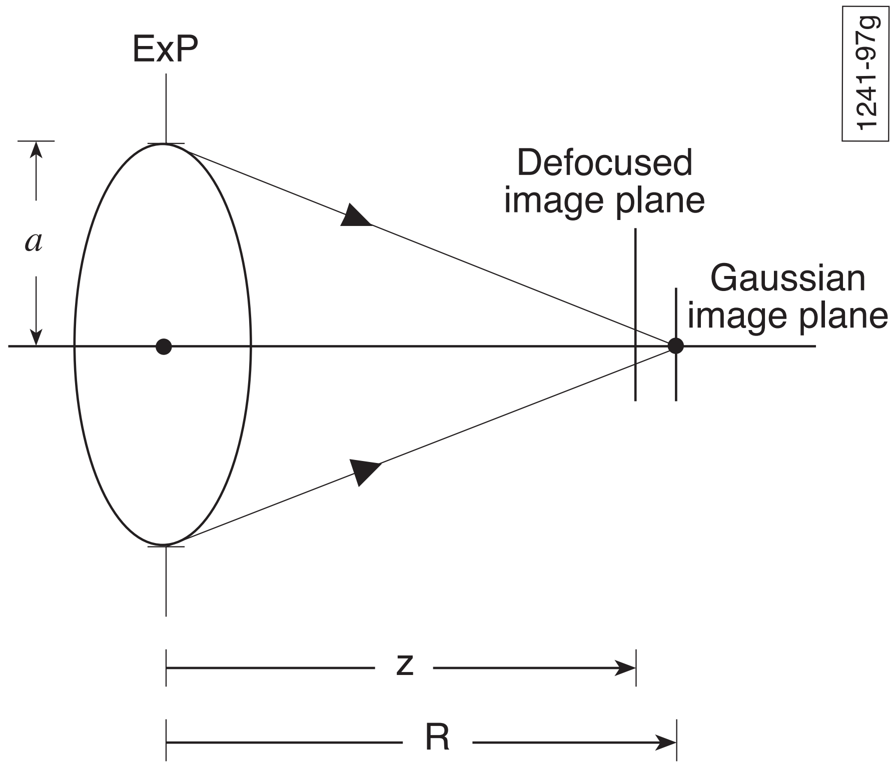
Adjunct Professor
College of Optical Sciences
University of Arizona

virendra.n.mahajan@aero.org

Summary

- Defocus Aberration
- Defocused PSFs
- Axial Irradiance
  - Aberration-Free Beam
  - Aberrated Beam
- Depth of Focus
  - Photographic Camera
  - Laser Transmitter
- Collimated Beam and Far-Field Distance
- Fraunhofer and Fresnel Diffraction
- Design of a Pinhole Camera

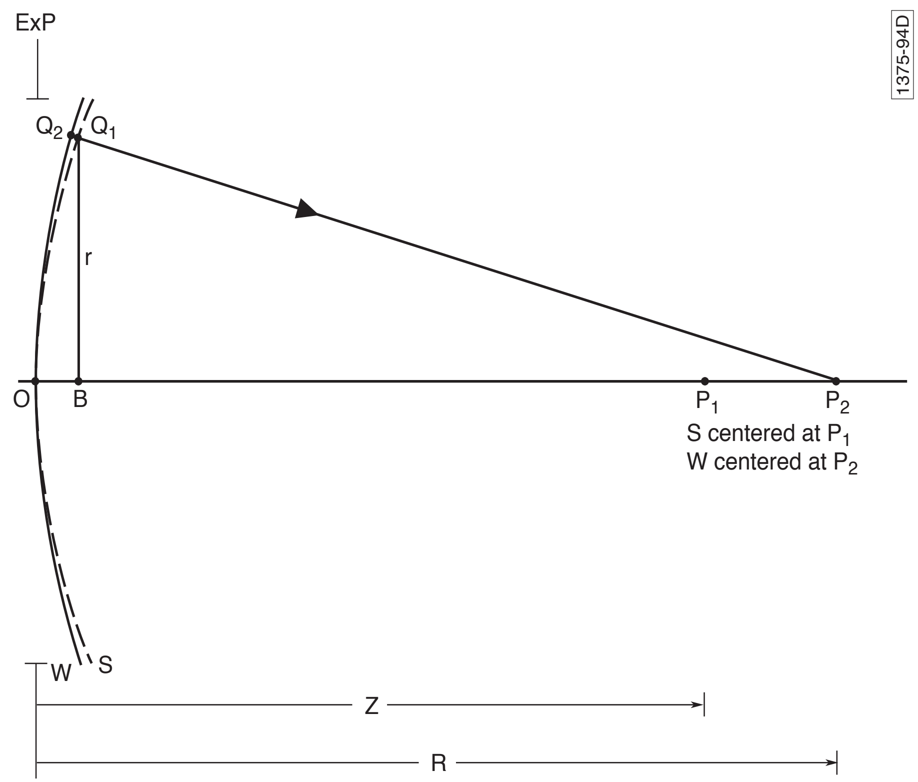
Beam Focusing and Longitudinal Defocus



$$I(r;z) = \left(\frac{2R}{z}\right)^2 \left| \int_0^1 \exp[i\Phi_d(\rho)] J_0\left(\pi \frac{R}{z} \rho r\right) \rho \, d\rho \right|^2 \quad \text{(2-81)}$$

- Irradiance in units of its aberration-free central value $P_{ex} S_{ex} / \lambda^2 R^2$.

Defocus Wave Aberration



• Spherical wavefront W yields an aberration-free image in the Gaussian image plane at a distance R.

• A defocused image is formed when observed in a plane at a distance z.

• Aberrations reduce the central irradiance and spread light from the Airy disc into the diffraction rings.

• Defocus **wave** aberration is approximately equal to the difference in the sags of two spheres with radii of curvature z and R:

$$W(r) = \frac{1}{2}\left(\frac{1}{z} - \frac{1}{R}\right)r^2$$

Letting $\rho = r/a$, $0 \le \rho \le 1$

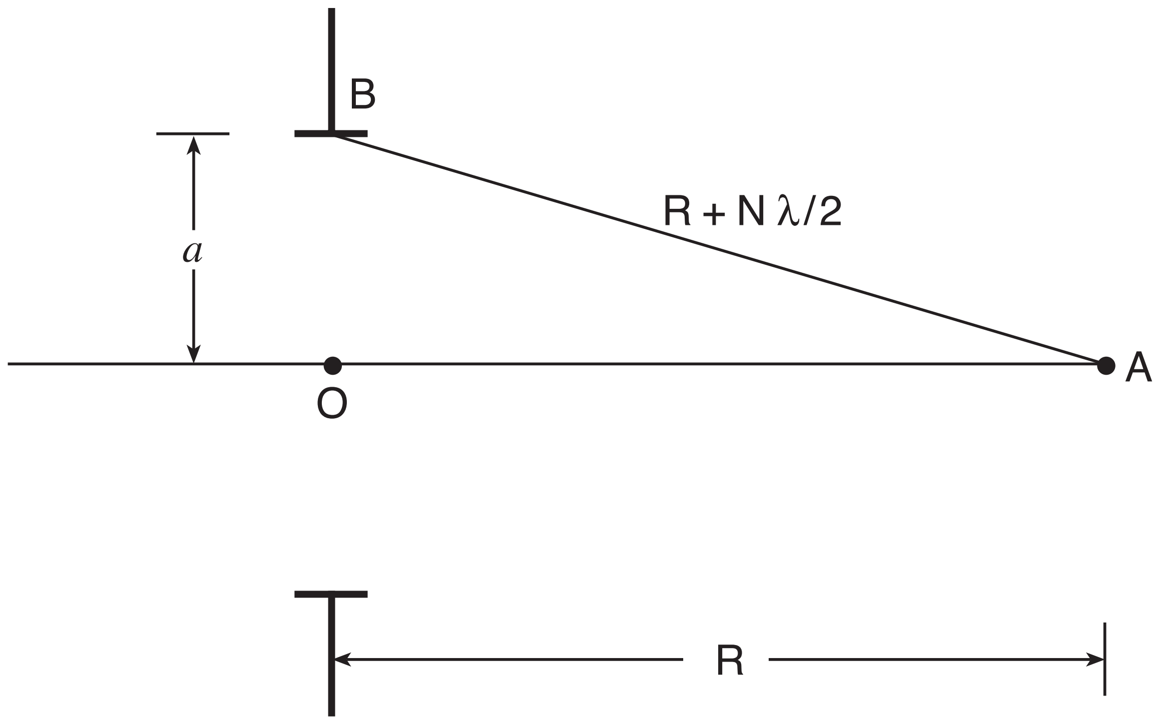
$$W(\rho) = B_d \rho^2 \quad , \quad B_d(z) = \frac{1}{2}\left(\frac{1}{z} - \frac{1}{R}\right)a^2$$

• Peak defocus **phase** aberration

$$B_d(z) = \frac{\pi a^2}{\lambda R}\left(\frac{R}{z} - 1\right) \quad , \quad \Phi_d(\rho) = B_d \rho^2$$

• Let us write it in terms of the Fresnel number N

Fresnel number N of the pupil as observed from a point A representing the number of Fresnel's half wave zones in the pupil



$$(R + N\lambda/2)^2 = R^2 + a^2$$

$$R^2 + NR\lambda = R^2 + a^2 \Rightarrow N = \frac{a^2}{\lambda R}$$

• Peak defocus **phase** aberration in terms of the Fresnel number N

$$B_d = \pi N\left(\frac{R}{z} - 1\right) \quad , \quad N = \frac{a^2}{\lambda R}$$

• **Large** Fresnel number, **small** depth of focus

• Photographic camera

• **Small** Fresnel number, **large** depth of focus

• Laser transmitter

Defocused PSF:

$$I(r;z) = \left(\frac{2R}{z}\right)^2 \left| \int_0^1 \exp[i\Phi_d(\rho)] J_0\left(\pi \frac{R}{z}\rho r\right) \rho \, d\rho \right|^2 \quad \text{(2-81)}$$

• $B_d = 0$ yields the Airy pattern $[2J_1(\pi r)/\pi r]^2$

• **For large values of N**, B_d becomes significant even for small differences between z and R. Hence, the system has a **small depth of focus**. Let $\Delta = z - R$, and since $z \simeq R$**,**

$$B_d(z) = \frac{1}{2}\left(\frac{1}{z} - \frac{1}{R}\right)a^2 = \frac{1}{2}\left(\frac{R-z}{zR}\right)a^2 \simeq -\frac{\Delta}{2}\left(\frac{a}{R}\right)^2 = -\frac{\Delta}{8F^2}$$

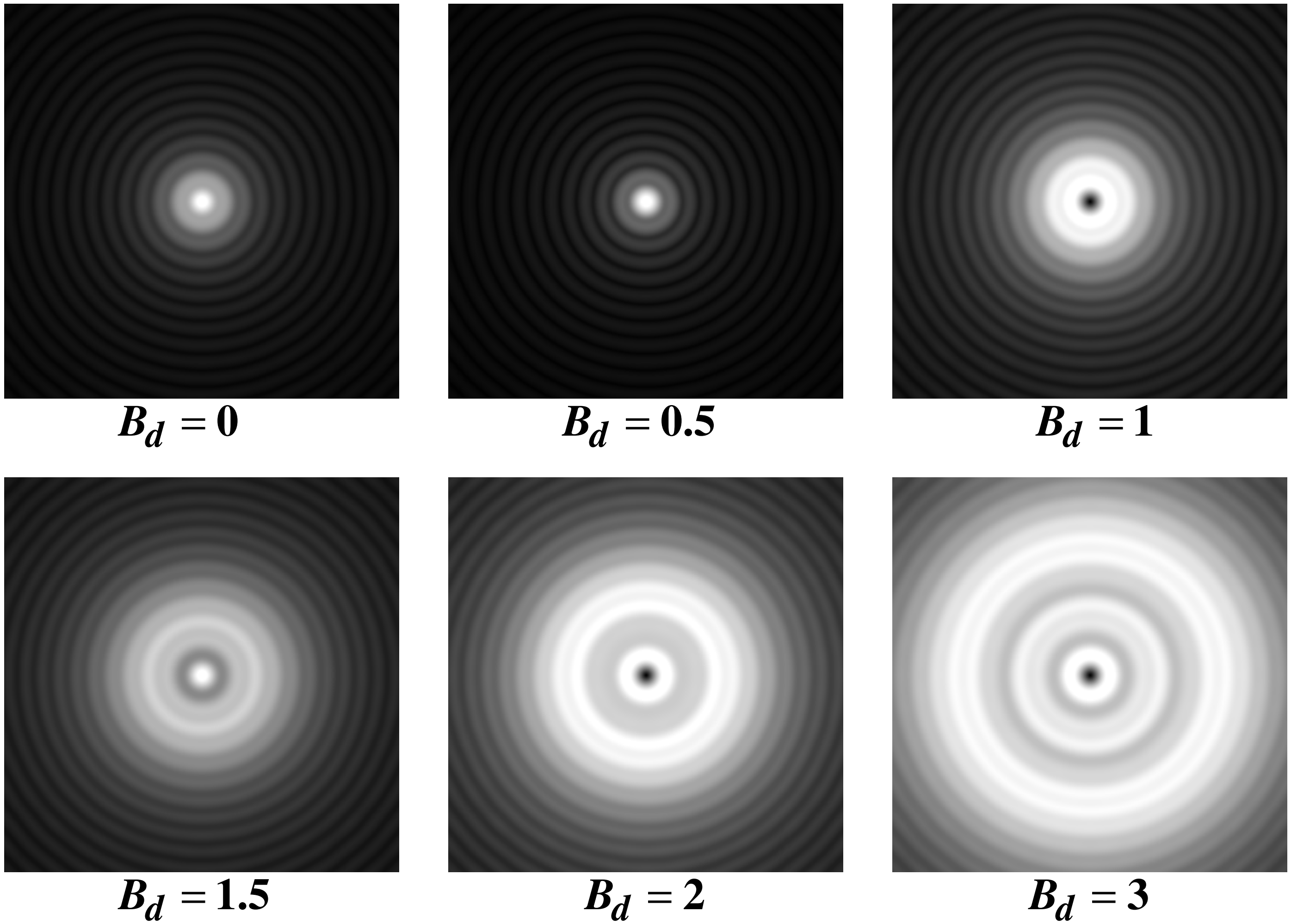
$$B_d(R+\Delta) = -\frac{\Delta}{8F^2} = -B_d(R-\Delta)$$

$$I(r;z) = 4\left|\int_0^1 \exp\left(iB_d\rho^2\right)J_0(\pi\rho r)\,\rho\,d\rho\right|^2$$

$$I(r;R+\Delta) = I(r;R-\Delta)$$ (Δ is longitudinal defocus)

• Defocused PSF is *symmetric* about the Gaussian image plane (for large N).

Defocused PSFs for Large Fresnel Numbers



- Dark spot at the center is obtained for integral number of waves of defocus

Strehl Ratio

$$S = \frac{\text{Axial irradiance at a distance } z, \text{ beam focused at } R}{\text{Axial irradiance at a distance } z, \text{ beam focused at } z} \leq 1$$

• S represents the effect of defocus only as an aberration, since the effect of inverse-square law is the same in both cases.

V. N. Mahajan, "Axial irradiance and optimum focusing of laser beams," Appl. Opt. **22**, 3042–3053 (1983).

Axial Irradiance of an "Aberration-Free" Beam

$$I(0;z) = \left(\frac{R}{z}\right)^2 \left(\frac{\sin B_d/2}{B_d/2}\right)^2$$

Depends on two competing factors:

1. **Increase** due to inverse-square law dependence for $z < R$.

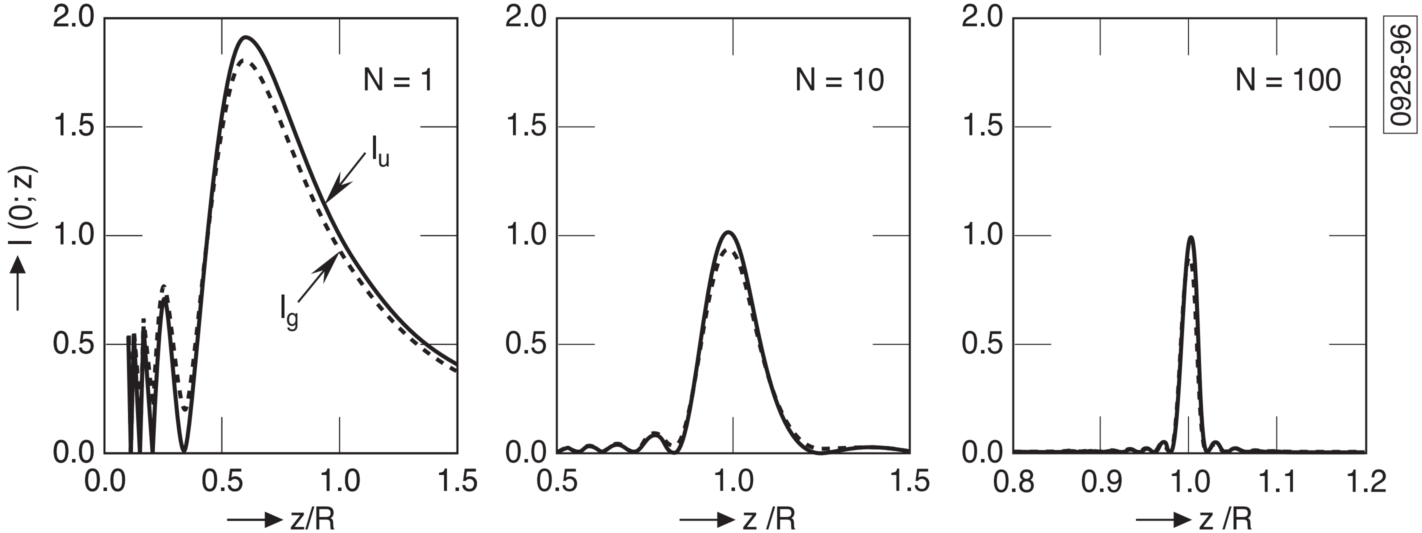
   • Significant effect only for large depth of focus or small N

2. **Decrease** due to defocus aberration (nonconstructive interference of Huygens' spherical wavelets)

   • Defocus aberration reduces irradiance regardless of the value of N

• Axial irradiance closer to the pupil can be higher than the focal-point irradiance if the increase due to the inverse square law is higher than the decrease due to defocus aberration.

Axial irradiance for different values of Fresnel number *N*



For $N = 1$, $z_p = 0.6R$, $|W_m| \equiv B_d = \lambda/3$

$$S = \left(\frac{\sin B_d/2}{B_d/2}\right)^2 = 0.68 \;,\; \left(\frac{R}{z_p}\right)^2 = \left(\frac{1}{0.6}\right)^2 = 2.78$$

- $I(z_p) = 1.9\, I(R)$ (Dashed curves are for an e^{-2}-truncated Gaussian beam)
- **Large (small) depth of focus for small (large) *N*.**
- **Asymmetry for large *N***

Optimum focusing of a beam on a target at a given distance z

• Since the axial irradiance peaks at a point closer to the focusing pupil, we ask where we should focus the beam for maximum central irradiance on the target? On the target or beyond the target?

• Since the target distance is fixed, the inverse-square law dependence is fixed.

• Hence, the irradiance on the target decreases (due to nonconstructive interference of Huygens' spherical wavelets) if the beam is not focused on it.

• Accordingly, a beam focused on the target yields maximum central irradiance on it, even though a larger value occurs closer to the pupil when N is small.

• **We check by focusing a beam at various distances (R varies, z fixed).**

Central irradiance on a target

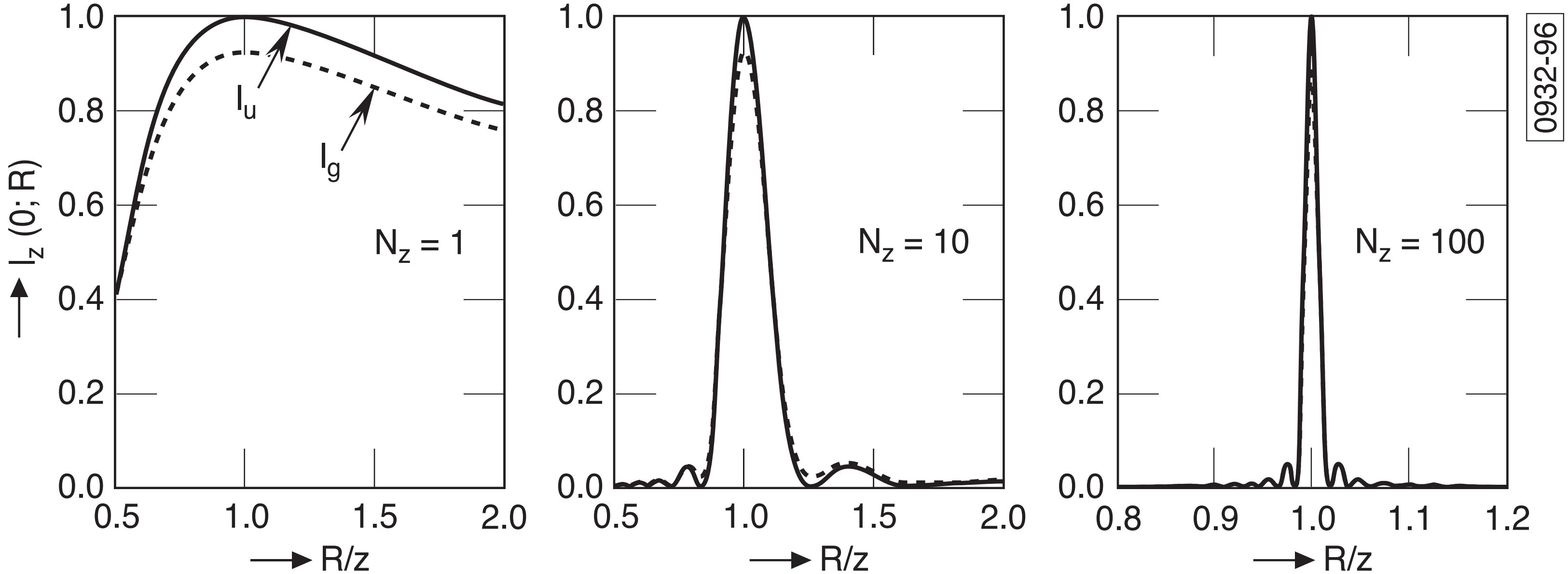


Figure 2-13. Central irradiance on a target at a fixed distance z from the plane of the pupil when a beam is focused at various distances R. The quantity $N_z = a^2/\lambda z$ represents the Fresnel number of the pupil as observed from the target. (Dashed curves are for an e^{-2}-truncated Gaussian beam.)

• Maximum central irradiance on the target is obtained when the beam is focused on it, i.e., when $R = z$.

Axial Irradiance of an Aberrated Beam

• How is the axial irradiance affected by an aberration?

• Yoshida and Asakura (1996) showed that the peak irradiance of a weakly-truncated focused Gaussian beam is higher when spherical aberration is introduced, and referred to this result as **"beyond the conventional diffraction limit."**

• Similarly, Jiang and Stamnes (1997) showed that the peak irradiance of a uniform focused beam is higher when spherical aberration is introduced, and characterized the result as **"quite unexpected."**

• However, these results are neither "beyond the conventional diffraction limit" nor "unexpected."

• They are a consequence of balancing of defocus aberration with spherical aberration.

• Since aberrations reduce the central irradiance, the (incorrect) expectation is that they also reduce the axial irradiance.

• Diffraction limit simply implies that the focal-point irradiance of a focused beam is maximum when it is aberration free.

• A beam observed in a plane other than the focal plane is not aberration free; it is in fact aberrated by the defocus aberration.

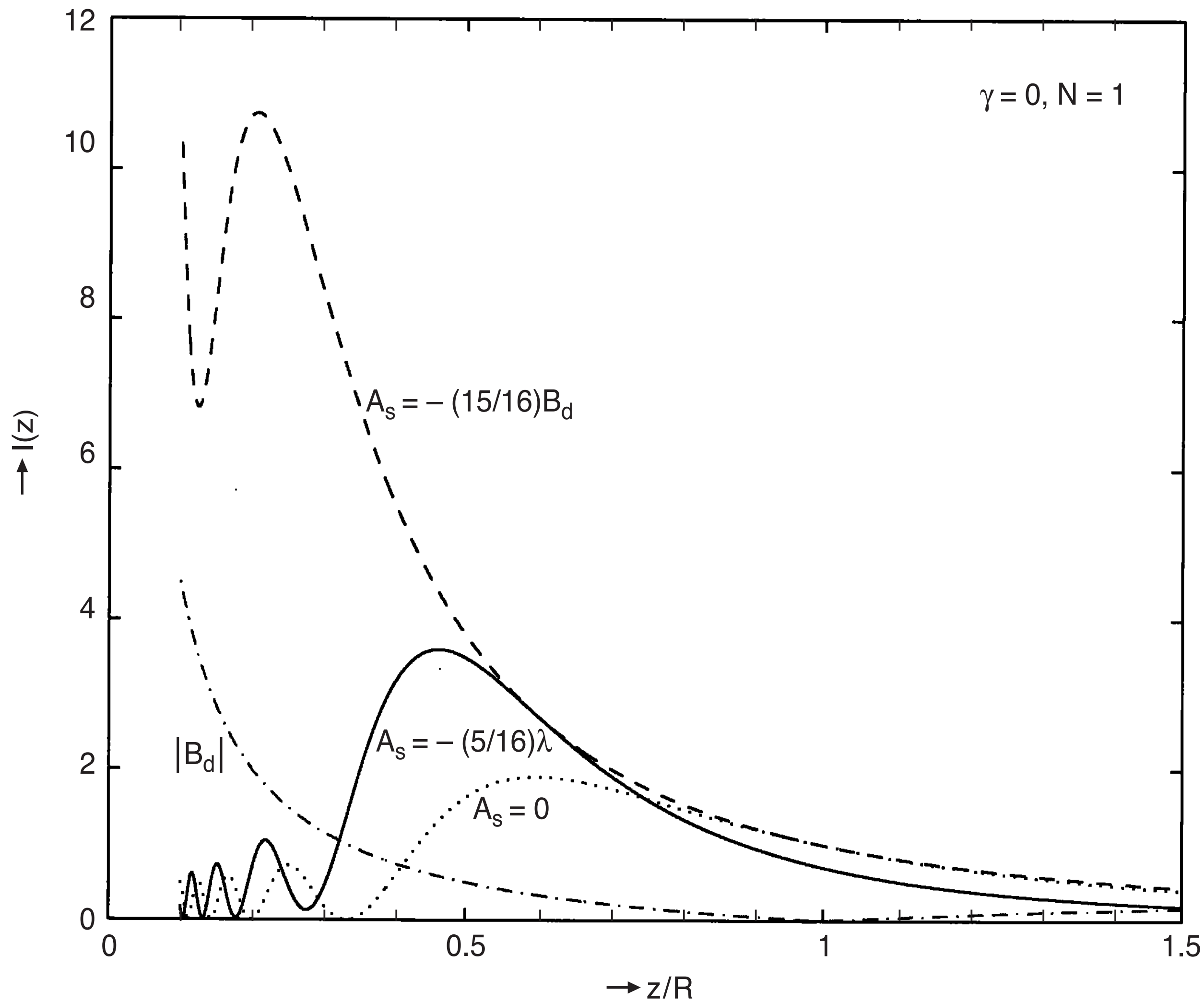
• Axial irradiance can be increased if spherical aberration or astigmatism is introduced to balance the defocus aberration.

• Defocus aberration: $B_d \rho^2$

• Balanced defocus aberration: $W_{bd}(\rho) = B_d\left(\rho^2 - \frac{15}{16}\rho^4\right)$

• Balancing decreases the standard deviation by a factor of 4 and, in turn, increases the axial irradiance.

Axial Irradiance With Spherical Aberration



V. N. Mahajan, “Axial irradiance of a focused beam,” J. Opt. Soc. Am. A **22**, 1814-1823 (2005).

Depth of Focus: Photographic Camera

$$D = 2 \text{ cm} \quad , \quad R = 10 \text{ cm} \quad , \quad \lambda = 0.5\ \mu\text{m}$$

$$F = \frac{R}{D} = 5 \quad , \quad N = \frac{a^2}{\lambda R} = 2000$$

$$S = \left(\frac{\sin B_d/2}{B_d/2} \right)^2 = 0.8 \Rightarrow B_d = \pm \frac{\lambda}{4}$$

$$z - R = -8F^2 B_d = \mp 25\ \mu\text{m}$$

• Depth of focus ($\pm 25\ \mu$m) is so small compared to the image distance of 10 cm that there is a negligible impact of the inverse-square-law dependence on the irradiance in the region of interest.

Depth of Focus: Laser Transmitter

$$D = 25\text{ cm} \quad , \qquad R = 1.5\text{ km} \quad , \qquad \lambda = 10.6\ \mu\text{m}$$

$$N = \frac{a^2}{\lambda R} = 1 \qquad , \qquad B_d = \pi N\left(\frac{R}{z} - 1\right)$$

$$S = \left(\frac{\sin B_d/2}{B_d/2}\right)^2 \quad \geq 0.8 \quad \text{for} \quad 1\text{ km} \leq z \leq 3\text{ km}$$

$\Rightarrow$ large depth of focus

• B_d becomes significant only when z is much different from R, implying a large depth of focus.

• Principal maximum lies at $z = 0.6R = 0.9\text{ km}$ with a value of 1.9 times the focal-point irradiance.

• Hence, the actual depth of focus is even larger.

Collimated Beam and the Far-Field Distance

• A collimated beam is equivalent to a beam focused at infinity.

• Hence, results for it can be obtained from those for a focused beam by letting $R \to \infty$, or Fresnel number $N = a^2 / \lambda R \to 0$.

• Since the collimated beam with a plane wavefront is observed at a distance z, the reference sphere has a radius of curvature z.

• Defocus phase aberration at a distance z represents the deviation of a planar wavefront from the reference sphere:

$$B_d = \frac{\pi}{\lambda}\left(\frac{1}{z} - \frac{1}{R}\right)a^2 = \pi N\left(\frac{R}{z} - 1\right)$$

$$= \frac{\pi a^2}{\lambda z} = \frac{S_{ex}}{\lambda z} \qquad \text{for} \qquad R \to \infty$$

• **Axial irradiance:**

$$I(0;z) = 4I_0 \sin^2\left(\pi a^2 / 2\lambda z\right) \quad , \quad I_0 = P_{ex}/S_{ex}$$ (Pupil irradiance)

• **Maxima:**

$$I(0;z) = 4I_0$$ at $$z = a^2/\lambda\,(2n+1) \quad , \quad n = 0, 1, 2, \ldots; \qquad z_{\text{largest}} = a^2/\lambda$$

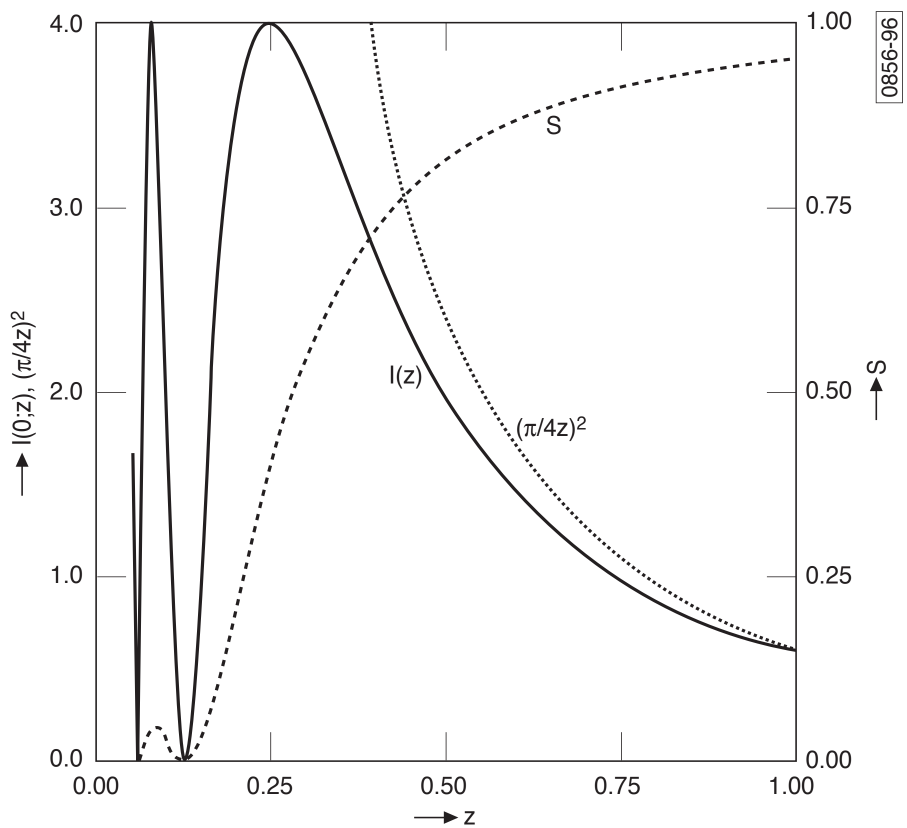
• Fresnel number $N_z = a^2/\lambda z$ observed from a distance z is $2n+1$, which is odd and yields a maximum of irradiance.

• z values decrease as n increases.

• **Minima:**

$$I(0;z) = 0$$ at $$z = a^2/2\lambda\,(n+1) \quad , \quad n = 0, 1, 2, \ldots; \qquad z_{\text{largest}} = a^2/2\lambda$$

• Even number $2\,(n+1)$ of Fresnel zones; hence zero irradiance.



$$I(0;z) \;=\; 4\,I_0 \sin^2(\pi/8z) \quad , \quad z \text{ in units of } D^2/\lambda \quad , \quad P_{ex}S_{ex}/\lambda^2 z^2 \rightarrow (\pi/4z)^2$$

Figure 2-15. Axial irradiance of a **collimated beam** at a distance z, normalized by the exit pupil irradiance I_0, compared with that of a corresponding focused beam. Ratio of the two represents the Strehl ratio S. At $z = 1$, $S = 4\left[\sin^2(\pi/8)\right]/(\pi/4)^2 = 0.95$.

Key observations:

- For $z > a^2/\lambda$, the axial irradiance decreases monotonically to zero.

- For $z \geq D^2/\lambda$, it decreases approximately as z^{-2}.

- At $z = D^2/\lambda$, $B_d = \pi/4$ or $\lambda/8$, and a collimated beam gives an axial irradiance that is 0.95 times the irradiance at this point if the beam were focused at it, i.e., $S = 0.95$.

- A collimated beam yields practically the same irradiance on a target lying in the *far field* $z \geq D^2/\lambda$ *of the exit pupil* as a beam focused on it; in other words, beam focusing does not significantly increase the power concentration on the target.

- $z = D^2/\lambda$ is called the **far-field distance** of a pupil.

Radially symmetric PSF:

$$I(r;z) = \left(\frac{2R}{z}\right)^2 \left| \int_0^1 \exp\left(iB_d\rho^2\right) J_0\left(\pi \frac{R}{z} \rho r\right) \rho \, d\rho \right|^2 \quad \text{(2-81)}$$

• Irradiance in units of its aberration-free central value $P_{ex}S_{ex}/\lambda^2 R^2$ and r in units of $\lambda F = \lambda R/D$.

• For a **collimated beam**, $B_d = S_{ex}/\lambda z$. We write irradiance in units of $P_{ex}S_{ex}/\lambda^2 z^2$ and r in units of $\lambda z/D$. Thus,

$$I(r;z) = 4 \left| \int_0^1 \exp\left(iB_d\rho^2\right) J_0(\pi\rho r) \rho \, d\rho \right|^2 \quad \text{(2-95)}$$

• Irradiance distribution represents the *Fresnel or the near-field diffraction pattern* of a circular pupil.

• For $z \geq D^2/\lambda$, defocus aberration $B_d \leq \pi/4$ or $\lambda/8$ is negligible.

$$\therefore \quad I(r;z) \simeq 4\left|\int_0^1 J_0(\pi\rho r)\,\rho\,d\rho\right|^2 = \left[2J_1(\pi r)/\pi r\right]^2 \quad , \quad z \geq D^2/\lambda$$

$$P(r_c) = \left[1 - J_0^2(\pi r_c) - J_1^2(\pi r_c)\right] \quad (r \text{ and } r_c \text{ in units of } \lambda z/D)$$

• $z \geq D^2/\lambda$ is called the far-field condition, and the corresponding irradiance distribution is called the *Fraunhofer or the far-field diffraction pattern* of a circular pupil

• Difference between the Fresnel and Fraunhofer diffraction patterns is the effect of the defocus aberration.

• Hence, Fresnel diffraction may be considered as defocused Fraunhofer diffraction.

Design of a Pinhole Camera

• A pinhole camera (or camera obscura) has been the subject of many investigations, including those by Petzval and Rayleigh.

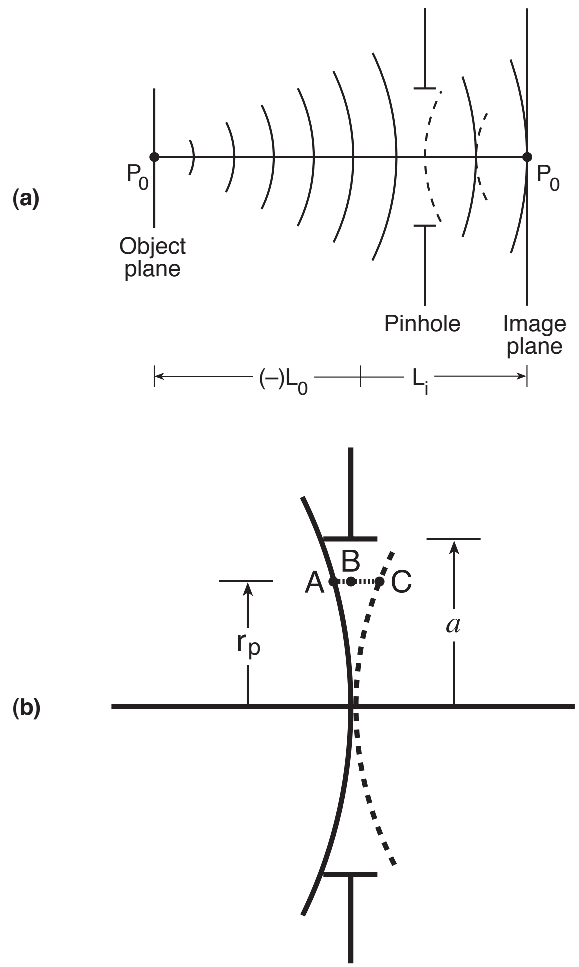
• It is simple (a pinhole of radius a on one side of a box of length L and the film on the other), *distortion free* with an *infinite depth of field* and a *very wide field of view*.

Design by using the defocus aberration tolerance approach:

• Difference between a pinhole camera and a regular camera is that the former does not use a lens to form the image.

• Lens in a regular camera converts a diverging spherical wave from a point object P_0 into a spherical wave converging to an image point P_0' in the image plane.

• Without a lens, a spherical wave of radius of curvature L_o diverging from the object P_0 is incident on the pinhole and continues as a diverging wave toward the image plane at a distance L_i and forms a defocused image at P_0'.



(a) Imaging by a **pinhole camera** of radius a. (b) Wavefront incident on the pinhole and emerging wavefront (shown shaded) required for perfect imaging. The pinhole size is extremely exaggerated for clarity of the wavefronts. The camera length $L_i >> a$.

• Defocus wave aberration represents the sum of the sags of two spherical wavefronts passing through the center of the pinhole with their centers of curvature lying at the object and image points:

$$AB + BC = AC$$

or $$W(r_p) = \frac{1}{2}\left(\frac{1}{L_i} - \frac{1}{L_o}\right) r_p^2$$

• r_p is the radial distance of a point B in the plane of the pinhole from its center (L_o is numerically negative according to our sign convention).

• Image will be practically diffraction limited according to the *Rayleigh's quarter wave rule, if the peak value of defocus wave aberration is less than or equal to* $\lambda/4$.

$$B_d = \frac{1}{2}\left(\frac{1}{L_i} - \frac{1}{L_o}\right) a^2 = \frac{\lambda}{4}$$

or $$\frac{1}{L_i} - \frac{1}{L_o} = \frac{\lambda}{2a^2} = \frac{1}{f_e}$$

• $f_e = 2a^2/\lambda$ is the effective focal length of the pinhole.

• For a distant object ($L_o \rightarrow -\infty$), we obtain

$$B_d = \frac{a^2}{2L_i} = \frac{\lambda}{4} \Rightarrow a = \sqrt{\lambda L_i/2}$$

• Same result as obtained by Petzval in 1857 by minimizing the spot radius resulting from the sum of the geometrical and diffraction contributions.

• Image spot for a point object is approximately the Airy disc with a radius of $0.61\lambda L_i/a = 0.61\sqrt{2\lambda L_i}$, or 0.61 times the value estimated by Petzval.

For additional reading:

V. N. Mahajan, “Axial irradiance of a focused beam,” *J. Opt. Soc. Am.* A **22**, 1814-1823 (2005).

V. N. Mahajan, “Strehl ratio of a focused beam,” *J. Opt. Soc. Am.* A **22**, 1824-1833 (2005).

A. Yoshida and T. Asakura, “Propagation and focusing of Gaussian laser beams beyond the conventional diffraction limit,” *Opt. Commun*. **123**, 694-704 (1996).

D. Y. Jiang and J. J. Stamnes, "Focusing at low Fresnel numbers in the presence of cylindrical or spherical aberration," *Pure Appl. Opt*. 6, 85-96 (1997).

V. N. Mahajan, Optical Imaging and Aberrations Part II: Wave Diffraction Optics (2001, SPIE Press, 2nd Printing, 2004).