Erasmus Mundus

Zernike polynomials and beyond

A workshop of the Erasmus Mundus Master course

« Optics in Science and Technology »

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Acknowledgements:
Société Française d'Optique,
Anciens de SupOptique

A workshop for advanced students

… and all those interested

- One **introductory** lecture (summary of M1 lectures in the Institut d'Optique master of optical engineering cursus)
- Five **advanced lectures** by guest lecturers
- **Poster** presentations on related issues (teaching, instruments, methods, research results)

Introductory lecture

Introduction to Zernike circle polynomials for the description of wavefronts and aberrations.

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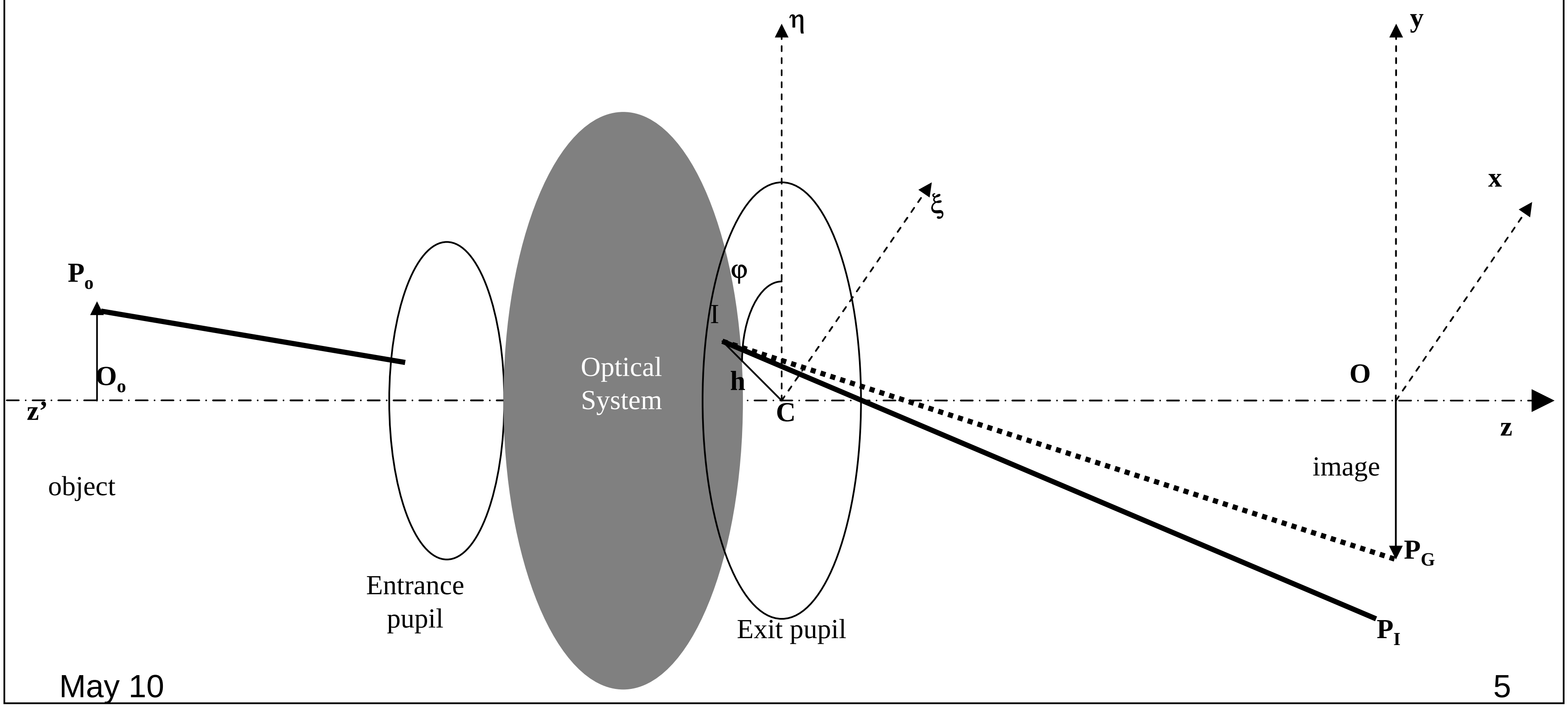
Outline

- The need for aberration expansions: system and wavefront parameters
- The Rayleigh-Sommerfeld diffraction integral for aberrated systems
- Aberration classification
  - Seidel
  - Zernike polynomials
  - Orthonormal wavefront expansion
- Applications:
  - 3rd order aberrations
  - Adaptive optics

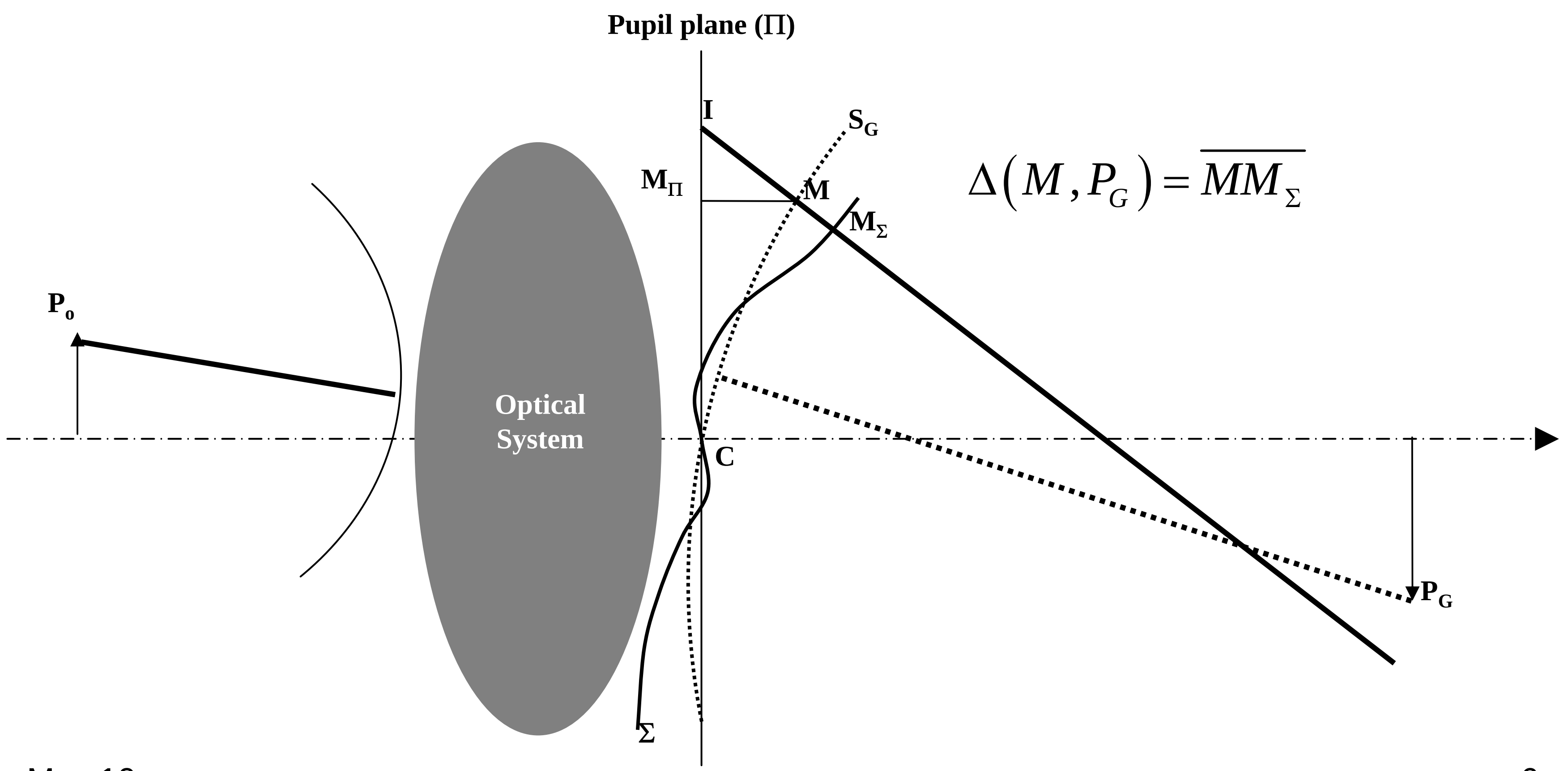
System description

Consider centered optical system, with identified Gaussian elements.

Arbitrary ray through point I of exit pupil intersects Gaussian image plane at P_I



Wavefront departure (path difference)

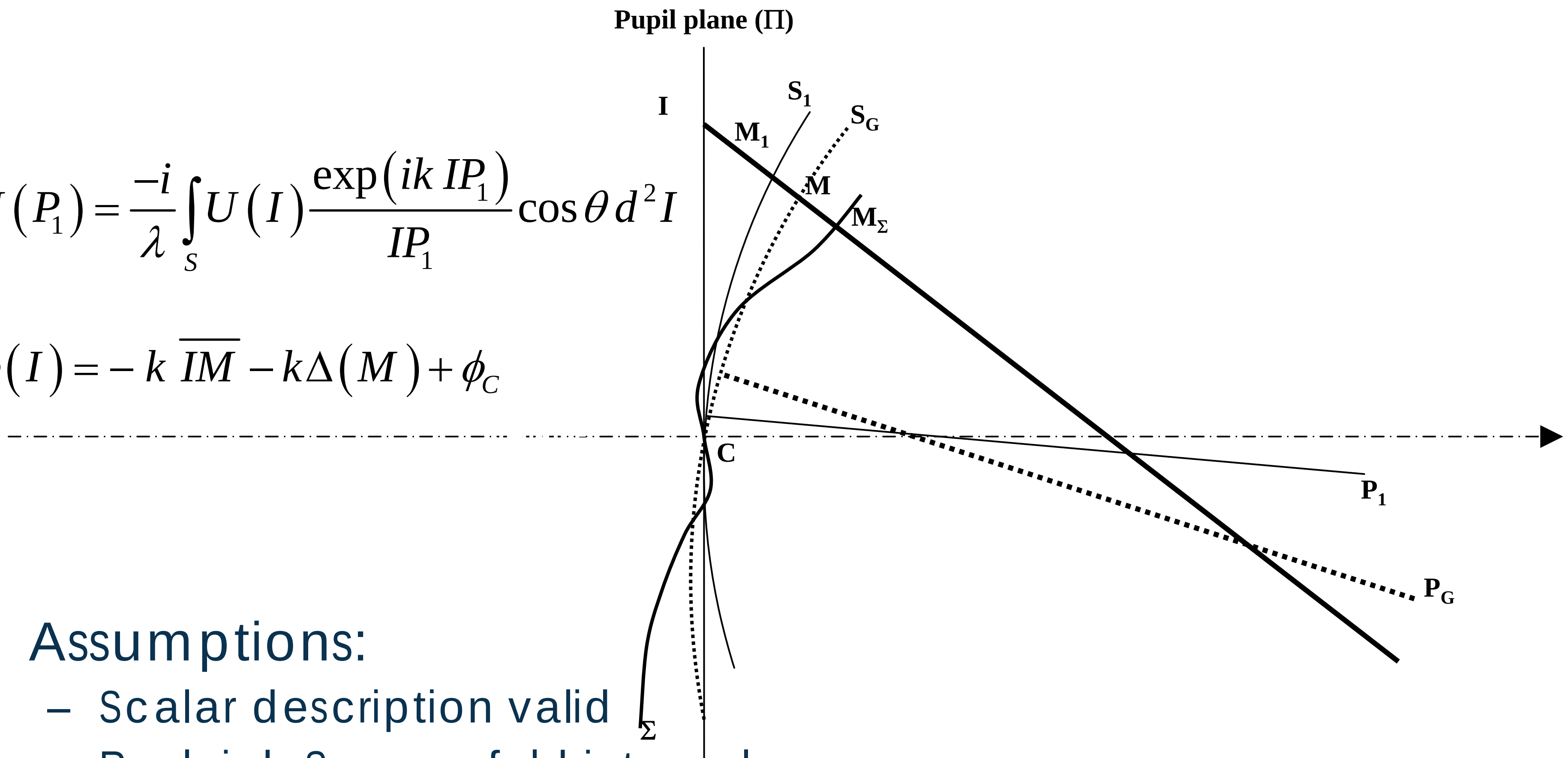


$$\Delta\left(M, P_G\right) = \overline{MM_\Sigma}$$

The diffraction integral

$$U(P_1) = \frac{-i}{\lambda} \int_S U(I) \frac{\exp(ik\, IP_1)}{IP_1} \cos\theta\, d^2 I$$

$$\phi(I) = -k\,\overline{IM} - k\Delta(M) + \phi_C$$

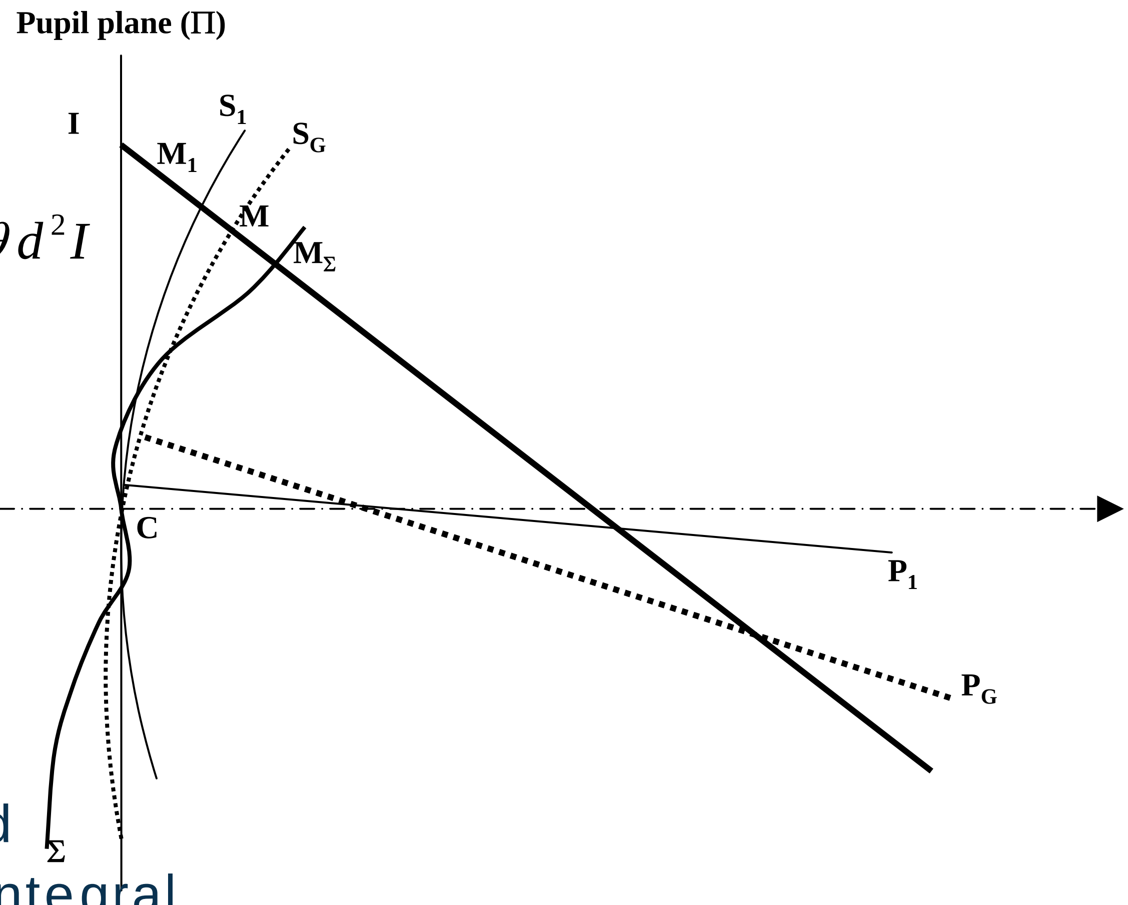


Phase

- Assumptions:
  - Scalar description valid
  - Rayleigh-Sommerfeld integral
  - Pupil uniformly illuminated

$$U(P_1) = \frac{-iA\exp(i\phi_C)}{\lambda} \int_{I\in\Pi} \frac{1}{IP_1} \exp\left[-ik\Delta(M) + ik\, MP\right] d^2 I$$

The diffraction integral



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- Assumptions:
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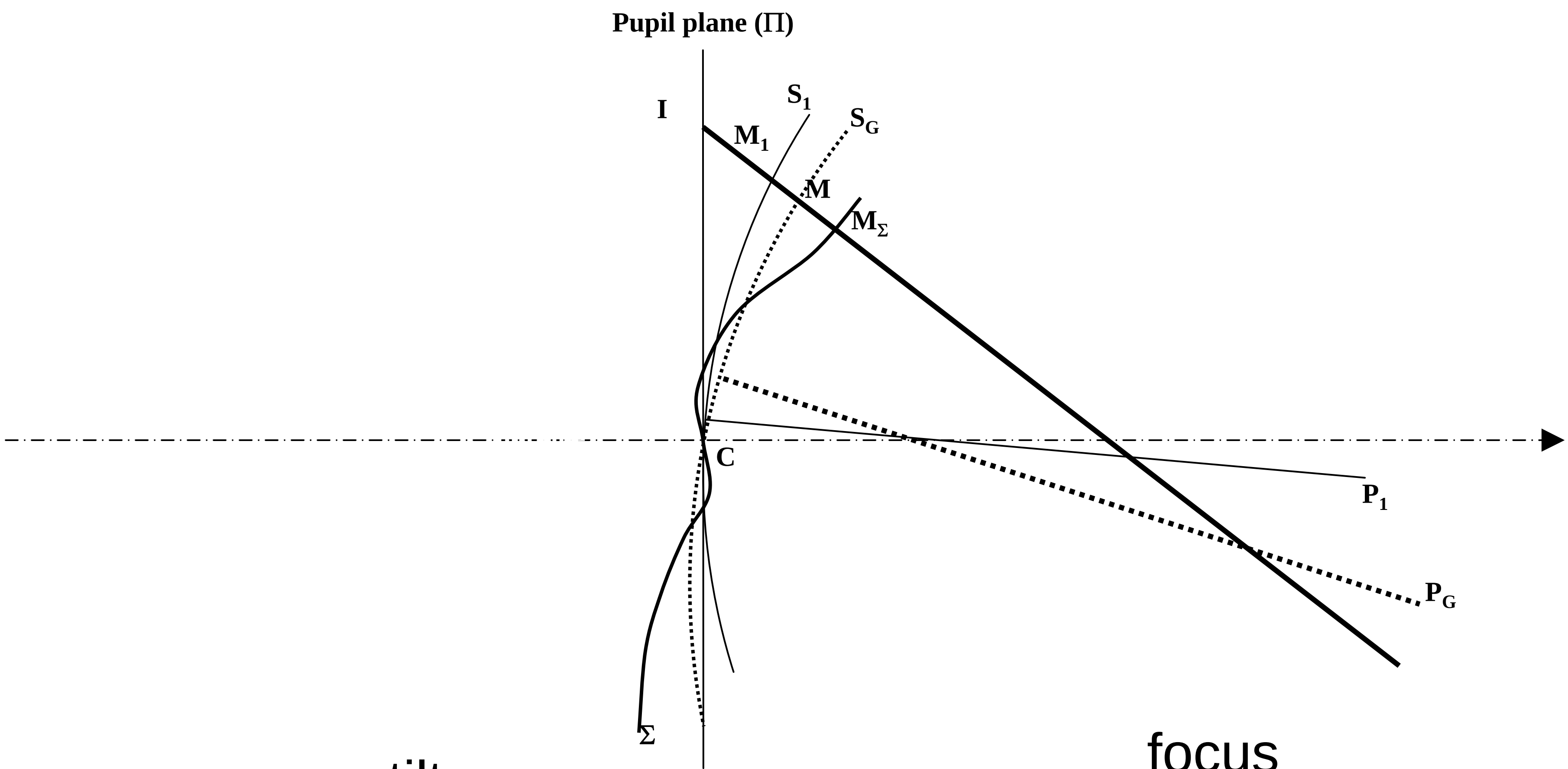
$$U(P_1) = \frac{-iA\exp(i\phi_C)}{\lambda}\int_{I\in\Pi}\frac{1}{IP_1}\exp\Big[\overbrace{-ik\Delta(M)+ik\,MP}^{\text{Phase }\psi}\Big]d^2 I$$

Change of reference sphere



tilt

focus

$$\overline{MM_1} = -\frac{\xi\,\delta x_1 + \eta\,\delta y_1}{R} - \frac{\varepsilon h^2}{2R^2}$$

Practical form

- Under the Fresnel approximation
- and if the denominator is essentially constant,

$$U\left(P_1\right)=\frac{-iAe^{i\phi_C+ik\left(R+\varepsilon\right)}}{\lambda R}$$

Pupil characteristic function

$$\iint_{\mathbb{R}^2} p_o\left(\xi,\eta\right)\exp\left(-ik\Delta\left(\xi,\eta\right)\right)\exp\left(-ik\varepsilon\frac{\xi^2+\eta^2}{2R^2}\right)$$

$$\exp\left(-ik\frac{\xi\,\delta x_1+\eta\,\delta y_1}{R}\right)d\xi\,d\eta$$

- the resulting amplitude is the Fourier transform of the phase term, focus included (FT of the aberrated pupil function).

Immediate consequences

•The center of the Airy disk is the absolute maximum of the irradiance that can be obtained in a given plane (z=R)

$$E_{MA} = \left(\frac{A\pi a^2}{\lambda R} \right)^2$$

•The image irradiance is an integral of a (space variant!) incoherent « point spread function » weighted by the geometrical image.

•Not rigorously an impulse response in the sense of linear, shift invariant systems.

$$E\left(\vec{r}\right) = \int E_{\text{geom}}\left(\vec{r_G}\right) \text{IPSF}\left(\vec{r} - \vec{r_G}, \vec{r_G}\right) d^2\vec{r_G}$$

•The « Strehl ratio » is the ratio of the irradiance at a given point to E_{MA}.

•Concurrent definitions exist.

$$R_S = \frac{E_M}{E_{MA}}$$

Small « aberrations »

$$\left|U\left(P_1\right)\right|^2=\left(\frac{A}{\lambda R}\right)^2\left|\iint_S e^{i\psi(M)}d^2M\right|^2=\left(\frac{A\pi a^2}{\lambda R}\right)^2\left|\frac{\int_0^1\int_0^{2\pi}\left(1+i\psi-\frac{\psi^2}{2}\right)u\,du\,d\theta}{\int_0^1\int_0^{2\pi}u\,du\,d\theta}\right|^2$$

$$=E_{MA}\left|1+i\overline{\psi}-\frac{\overline{\psi^2}}{2}\right|^2=E_{MA}\left(1-\sigma_\psi^2\right)$$

- Nijboer (1943) expansion for small phases: the local irradiance is directly given by the phase variance over the pupil
- Whence the Maréchal criterion (1944):
  - 20% loss for variance $\lambda^2/180$

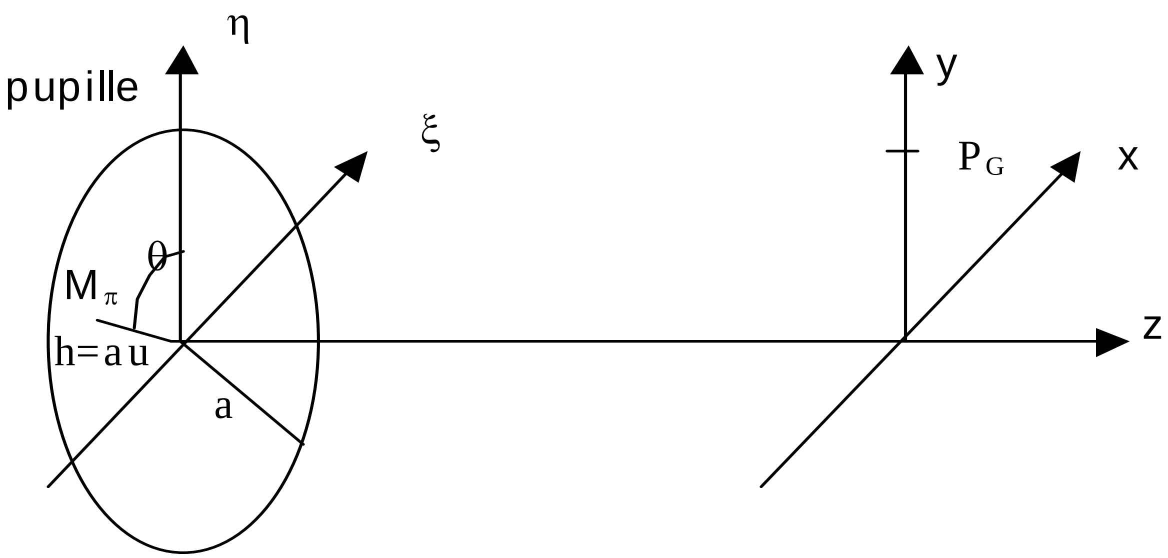
Aberration expansions

- To get a good feeling of the behaviour
  - of optical systems being designed
  - of wave propagation

an arbitrary phase function is not convenient.

- Here, we consider the important case where a small number of terms of an expansion onto a suitable basis is appropriate.

The Seidel aberrations



$$\Delta(u,y,\theta)=\sum_{(p,q,m)\in\mathbb{N}} A_{pqm}\,u^{2p+m}\,y^{2q+m}\cos m\theta$$

- In the most intuitive representation (Seidel 1855), all aberration terms are monomials, not polynomials.
- Aberration order n=2p+2q+2m-1

Orders 1 and 3

n	p	q	m	name
1	1	0	0	Defocus
	0	1	0	Vergence from focus
	0	0	1	Tilt
3	2	0	0	Primary spherical
	0	2	0	Primary piston
	0	0	2	Primary astigmatism
	1	1	0	Primary curvature
	1	0	1	Primary coma
	0	1	1	Primary distorsion

$$u^{2p+m}\ y^{2q+m}\cos m\theta$$

The Zernike circle polynomials

- Zernike, 1934: given the scalar product

$$\left(f,g\right)=\frac{\int_0^1\int_0^{2\pi} f\left(u,\theta\right)g^*\left(u,\theta\right)u\,du\,d\theta}{\pi}$$

a complete set of orthogonal polynomials $\mathfrak{R}_n^m\left(u\right)\mathrm{e}^{im\theta}$

of degree n can be defined such that:

$n\in\mathbb{N},\quad m\in\mathbb{Z}\ ,$

all n<m terms vanish

all terms with n-m odd vanish

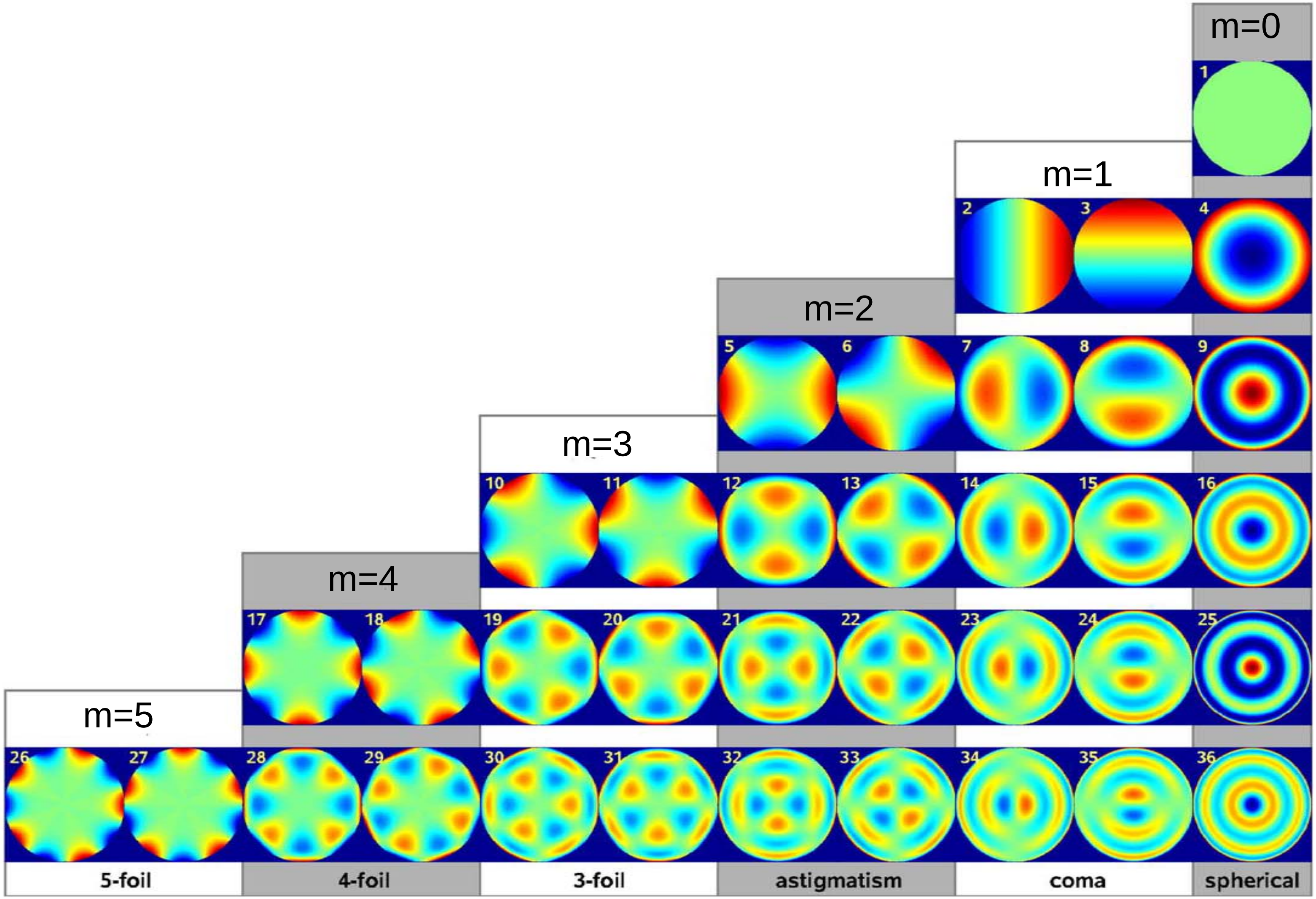
$\mathfrak{R}_n^m\left(0\right)=1$

The first Zernike polynomials

$\mathcal{R}_0^0(u) = 1$		$\mathcal{R}_2^0(u) = 2\,u^2 - 1$		$\mathcal{R}_4^0(u) = 6\,u^4 - 6\,u^2 + 1$
	$\mathcal{R}_1^1(u) = u$		$\mathcal{R}_3^1(u) = 3\,u^3 - 2\,u$	(higher order term)
		$\mathcal{R}_2^2(u) = u^2.$		

- All Seidel aberrations up to order 3 are recognized, but associated to other terms

Visualisation



Ruoff and Totzek, J. Micro/Nanolith. MEMS MOEMS 8 (2009)

Seidel aberrations, Zernike aberrations

- So, now, what do we call spherical aberration: the Seidel aberration or the Zernike aberration that includes the former as its highest order term?
- V.N. Mahajan: "A Zernike polynomial aberration may also be referred to as a "balanced" aberration. For example. the Zernike primary spherical aberration R^4_0 consists of a classical primary spherical aberration (u^4 term) optimally balanced with defocus (u^2 term) to minimize its variance. It may be called "balanced primary spherical aberrations"."

Orthonormal Zernike circle polynomials

$$Z_n^m(u,\theta)=\begin{cases}\sqrt{\dfrac{2(n+1)}{1+\delta_m^0}}\mathcal{R}_n^{|m|}(u)\cos m\theta \text{ for m} \geq 0\\ -\sqrt{2(n+1)}\mathcal{R}_n^{|m|}(u)\sin m\theta \text{ for m<0}\end{cases}$$

$$\Delta(u,y,\theta)=\sum_{(n,q)\in\mathbb{N}^2,m\in\mathbb{Z}} D_{mnq}Z_n^m(u,\theta)y^{2q+m}$$

- Small aberrations: $\dfrac{E(P_1)}{E_{MA}}=1-\dfrac{4\pi^2}{\lambda^2}\sum_{m,n\neq 0,0} D'^2_{mn}$

- There exists also a single index numbering scheme
- Various notations exist. An ISO standard (ISO14999) offers a consensus.

Rayleigh 20% limit

Primary aberration term	Maximum admissible departure	
	At paraxial image	At best focus
sphericity	0,24 λ	0,95 λ
coma	0,20 λ	0,60 λ
astigmatism	0,17 λ	0,17 λ
curvature	0,25 λ	(Airy)
distorsion	0,14 λ	(Airy)

Adaptive optics

- Measure wavefront!
- Include actuators in the setup such that $\delta_p(u,\theta) = V_p \sum_{(m,n)\in\mathbb{N}^2} d_{mnp} Z_n^m(u,\theta)$

- If aberrations can be compensated by the actuators to a good degree, then

$$R_S = 1 - \frac{4\pi}{\lambda^2} \sum_{m,n\neq 0,0} \left(\sum_p V_p d_{mnp} + D'_{mn} \right)^2 Z_n^m(u,\theta)$$

- Minimizing versus V_p:

$$\sum_{m,n} \left(\sum_{p'} V_{p'} d_{mnp'} + D'_{mn} \right) d_{mnp} = 0$$

NB: the Zernike are not necessarily the appropriate basis for such minimization, see lecture by J.M. Conan